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## Spanning trees on graphs and lattices in $d$ dimensions

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**Abstract.** The problem of enumerating spanning trees on graphs and lattices is considered. We obtain bounds on the number of spanning trees  $N_{ST}$  and establish inequalities relating the numbers of spanning trees of different graphs or lattices. A general formulation is presented for the enumeration of spanning trees on lattices in  $d \geq 2$  dimensions, and is applied to the hypercubic, body-centred cubic, face-centred cubic and specific planar lattices including the kagomé, diced, 4–8–8 (bathroom-tile), Union Jack and 3–12–12 lattices. This leads to closed-form expressions for  $N_{ST}$  for these lattices of finite sizes. We prove a theorem concerning the classes of graphs and lattices  $\mathcal{L}$  with the property that  $N_{ST} \sim \exp(nz_{\mathcal{L}})$  as the number of vertices  $n \rightarrow \infty$ , where  $z_{\mathcal{L}}$  is a finite non-zero constant. This includes the bulk limit of lattices in any spatial dimension, and also sections of lattices whose lengths in some dimensions go to infinity while others are finite. We evaluate  $z_{\mathcal{L}}$  exactly for the lattices we consider, and discuss the dependence of  $z_{\mathcal{L}}$  on  $d$  and the lattice coordination number. We also establish a relation connecting  $z_{\mathcal{L}}$  to the free energy of the critical Ising model for planar lattices.

### 1. Introduction

The enumeration of spanning trees on a graph or lattice is a problem of long-standing interest in mathematics [1–4] and physics [5–7]. Let  $G = (V, E)$  denote a connected graph (without loops) with vertex (site) and edge (bond) sets  $V$  and  $E$ . Let  $n = v(G) = |V|$  be the number of vertices and  $e(G) = |E|$  the number of edges. A spanning subgraph  $G'$  is a subgraph of  $G$  with  $v(G') = |V|$ , and a tree is a connected subgraph with no circuits. A spanning tree is a spanning subgraph of  $G$  that is a tree (thus  $e(G') = n - 1$ ). The degree of a vertex is the number of edges attached to it (often denoted by the coordination number or valence). A  $\kappa$ -regular graph is a graph with the property that each of its vertices has the same degree  $\kappa$ . For these and further related definitions see, e.g., [1–3].

Denote the number of distinct spanning trees of a graph  $G$  by  $N_{ST}(G)$ . This number can be enumerated in terms of standard graph-theoretic quantities. Two methods for doing this will be used here: (a) via the Laplacian matrix [1, 2, 4] and (b) as a special value of the Tutte polynomial [8–10]. For the first, we recall the definition that two vertices are adjacent if they are connected by an edge. The adjacency matrix  $A$  of  $G$  is an  $n \times n$  matrix whose elements are

$$A_{ij} = \begin{cases} 1 & \text{if sites } i \text{ and } j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases} \quad (1.1)$$

and the degree matrix  $\Delta$  of  $G$  is an  $n \times n$  diagonal matrix with elements

$$\Delta_{ij} = \kappa_i \delta_{ij} \tag{1.2}$$

where  $\kappa_i$  is the degree of site  $i$  and  $\delta_{ij}$  is the Kronecker delta function. Define the Laplacian matrix

$$Q = \Delta - A. \tag{1.3}$$

Here and throughout this paper, we use boldface to denote matrices of size  $n \times n$ .

Since the sum of the elements in each row (or column) of  $Q$  vanishes, one of the eigenvalues of  $Q$  is zero. Denote the remaining  $n - 1$  eigenvalues by  $\lambda_1, \dots, \lambda_{n-1}$ . Then two basic theorems in graph theory state that [1, 2, 4]

$$N_{ST}(G) = \text{any cofactor of } Q \tag{1.4}$$

$$= \frac{1}{n} \prod_{i=1}^{n-1} \lambda_i. \tag{1.5}$$

An elementary proof of the equivalence of (1.4) and (1.5) can be found in [11]. The Laplacian matrix  $Q$  is also known in the literature as the Kirchhoff matrix, or simply the tree matrix, which arose in the analysis of electric circuits [5]. The enumeration of  $N_{ST}$  has very recently been considered in [11] for finite hypercubic lattices and the square net embedded on non-orientable surfaces.

A second way to calculate  $N_{ST}(G)$  is as the special value

$$N_{ST}(G) = T(G, 1, 1) \tag{1.6}$$

of the Tutte polynomial of the graph  $G$  [8–10],

$$T(G, x, y) = \sum_{G' \subseteq G} (x - 1)^{k(G') - k(G)} (y - 1)^{c(G')} \tag{1.7}$$

where  $k(G)$  is the number of connected components and  $c(G)$  is the number of independent circuits in  $G$ , and the summation is over all spanning subgraphs  $G'$  of  $G$ . Here, we have  $k(G) = 1$  for connected graphs  $G$ , and it is clear that (1.7) leads to (1.6), since in the limit of  $x, y \rightarrow 1$  the only contributing terms in (1.7) are those of  $k(G') = k(G) = 1$  and  $c(G') = 0$ , namely, the spanning trees.

In physics one often deals with lattices. A lattice is regular if all sites are equivalent [12]. For a wide class of graphs, including lattices which may or may not be regular and strips of lattices, the number of spanning trees  $N_{ST}$  has asymptotic exponential growth

$$N_{ST}(G) \sim \exp(nz_{\{G\}}) \quad \text{as } n \rightarrow \infty. \tag{1.8}$$

Thus  $z_{\{G\}}$  provides a natural measure of the rate of growth, and is evaluated via

$$z_{\{G\}} = \lim_{n \rightarrow \infty} \frac{1}{n} \ln N_{ST}(G) \tag{1.9}$$

where  $\{G\}$  denotes the formal  $n \rightarrow \infty$  limit of a graph of type  $G$ . For lattices  $\mathcal{L}$  this is known as the bulk, or the thermodynamic, limit, and we denote  $z_{\{G\}}$  by  $z_{\mathcal{L}}$ . Closed-form expressions for  $z_{\mathcal{L}}$  have been obtained for the square, honeycomb and triangular lattices [6, 7]. Exact results have also been obtained for strips of regular lattices of finite widths and infinite length [13–15].

In this paper we consider spanning trees on general graphs  $G$  as well as lattices  $\mathcal{L}$  in  $d \geq 2$  dimensions. Specifically, we shall:

- (a) derive an exact relation between  $z_{\mathcal{L}}$  and  $z_{\mathcal{L}^*}$  for planar lattices  $\mathcal{L}$  and  $\mathcal{L}^*$  which are mutually dual;
- (b) obtain bounds on  $N_{ST}(G)$  and  $z_{\langle G \rangle}$ ;
- (c) establish the exponential growth  $N_{ST}(G) \sim \exp(nz_{\langle G \rangle})$  for a wide class of graphs;
- (d) present a general formulation for the enumeration of  $N_{ST}(\mathcal{L})$  and  $z_{\mathcal{L}}$ ;
- (e) enumerate  $N_{ST}(\mathcal{L})$  and evaluate  $z_{\mathcal{L}}$  exactly for a number of lattices in  $d \geq 2$ ;
- (f) analyse the dependence of  $z_{\mathcal{L}}$  on the spatial dimensionality and the coordination number of the lattice  $\mathcal{L}$ , including deriving an asymptotic expansion for  $d$ -dimensional hypercubic lattices; and
- (g) establish a relation connecting  $z_{\mathcal{L}}$  to the free energy of the critical Ising model for planar lattices  $\mathcal{L}$ .

Before proceeding, it is useful to review here the close connection of spanning trees with the Potts model in statistical mechanics. The partition function of a  $q$ -state Potts model at temperature  $T = 1/\beta$  on  $G$  is [16, 17]

$$Z(G, q, v) = \sum_{\{\sigma_i\}} e^{-\beta\mathcal{H}} \quad (1.10)$$

where  $-\beta\mathcal{H} = K \sum_{\langle ij \rangle} \delta_{\sigma_i \sigma_j}$  with  $\langle ij \rangle$  ranging over pairs of adjacent vertices in  $G$ . The summation is taken over  $\sigma_i = 1, 2, \dots, q$  and  $i = 1, 2, \dots, n$ . The partition function (1.10) can be written as [18, 19]

$$Z(G, q, v) = \sum_{G' \subseteq G} q^{k(G')} v^{e(G')} \quad (1.11)$$

where the summation is over all spanning subgraphs of  $G$  and

$$v = e^K - 1. \quad (1.12)$$

The expression (1.11) shows that  $Z(G, q, v)$  is a polynomial in  $q$  and  $v$  and enables one to generalize  $q$  from positive integers to real and complex values. The Potts model partition function on a graph  $G$  is related to the Tutte polynomial according to

$$Z(G, q, v) = (x - 1)^{k(G)} (y - 1)^n T(G, x, y) \quad (1.13)$$

with

$$x = 1 + q/v \quad y = v + 1 \quad (1.14)$$

so

$$q = (1 - x)(1 - y). \quad (1.15)$$

Thus,  $N_{ST}(G)$  can also be calculated in terms of the Potts partition function  $Z(G, q, v)$ . The values  $x = 1$  and  $y = 1$  in the evaluation (1.6) correspond to the limits  $q \rightarrow 0$ , and  $v \rightarrow 0$  with  $\lim_{\{q,v\} \rightarrow 0} (q/v) = 0$ .

For a planar graph  $G$  and its dual graph  $G^*$ , the Tutte polynomial possesses the duality relation [2, 9, 10]

$$T(G, x, y) = T(G^*, y, x). \quad (1.16)$$

An immediate consequence of (1.6) and (1.16) is that the number of spanning trees is the same for a planar graph  $G$  and its dual  $G^*$  (see, for example, [2]), namely,

$$N_{ST}(G) = N_{ST}(G^*). \quad (1.17)$$

Let  $n^*$  be the number of sites of  $G^*$ , given by the Euler relation

$$n^* = |E| - n + 1. \tag{1.18}$$

For planar lattices  $\mathcal{L}$  and its dual  $\mathcal{L}^*$ , it is convenient to introduce the ratio in the bulk limit

$$v_{\mathcal{L}} = \lim_{n \rightarrow \infty} (n^*/n) \tag{1.19}$$

satisfying

$$v_{\mathcal{L}} v_{\mathcal{L}^*} = 1. \tag{1.20}$$

Using (1.19) and (1.20), we obtain the relation

$$z_{\mathcal{L}^*} = z_{\mathcal{L}}/v_{\mathcal{L}} \tag{1.21}$$

relating  $z_{\mathcal{L}}$  and  $z_{\mathcal{L}^*}$ . As examples, it is readily verified that we have

$$v_{hc} = \frac{1}{2} \quad v_{kag} = 1 \quad v_{4-8-8} = \frac{1}{2} \tag{1.22}$$

where the subscripts denote the honeycomb, kagomé and 4–8–8 lattices, respectively (for a detailed discussion on classifications of planar lattices see, for example, [12, 20]). Applying (1.21) to the respective duals, namely, the triangular, diced and Union Jack lattices, we obtain the relations

$$z_{tri} = 2z_{hc} \quad z_{diced} = z_{kag} \quad z_{UJ} = 2z_{4-8-8}. \tag{1.23}$$

For a regular planar lattice involving a tiling of the plane with only one type of polygon, the dual lattice is also regular. This includes the square lattice, which is self-dual, and honeycomb and triangular lattices, which are mutually dual. In contrast, for a regular planar lattice involving a tiling of the plane with more than one type of polygon, the dual lattice is not regular since it does not have a uniform coordination number. For example, the kagomé and 4–8–8 lattices, which are regular, involve tilings with more than one type of polygon, and their duals, the diced and Union Jack lattices, are not regular. For non-regular lattices it is convenient to introduce an effective coordination number  $\kappa_{\text{eff}}$  defined as the average number of edges per vertex,

$$\kappa_{\text{eff}} = \lim_{n \rightarrow \infty} (2|E|/n). \tag{1.24}$$

Combining (1.18) and (1.24), we derive the relation

$$\kappa_{\text{eff}} = 2(1 + v_{\mathcal{L}}). \tag{1.25}$$

As examples, using (1.20) this yields  $\kappa_{\text{eff}} = 4$  and 6, respectively, for the diced and the Union Jack lattices. For  $\kappa$ -regular graphs we have  $\kappa_{\text{eff}} = \kappa$ .

**2. General bounds**

The determination of upper bounds on  $N_{ST}(G)$  is a problem of considerable interest in graph theory. A general upper bound is [21]

$$N_{ST}(G) \leq \frac{1}{n} \left( \frac{2|E|}{n-1} \right)^{n-1}. \tag{2.1}$$

For a  $\kappa$ -regular graph, this implies the upper bound

$$N_{ST}(G) \leq \frac{1}{n} \left( \frac{n\kappa}{n-1} \right)^{n-1}. \tag{2.2}$$

and hence

$$z_{\mathcal{L}} \leq \ln \kappa. \quad (2.3)$$

More generally, we shall also deal with lattices that are not regular but for which one can define an effective coordination number,  $\kappa_{\text{eff}}$  as in (1.24). For these, from (2.1), one has

$$z_{\mathcal{L}} \leq \ln \kappa_{\text{eff}}. \quad (2.4)$$

A stronger upper bound for  $\kappa$ -regular graphs with  $\kappa \geq 3$  is due to McKay, Chung and Yau [22, 23], who established rigorously that

$$N_{ST}(G) \leq \left( \frac{2 \ln n}{n \kappa \ln \kappa} \right) (C_{\kappa})^n \quad (2.5)$$

where

$$C_{\kappa} = \frac{(\kappa - 1)^{\kappa-1}}{[\kappa(\kappa - 2)]^{\kappa/2-1}}. \quad (2.6)$$

This leads to the upper bound

$$z_{\mathcal{L}} \leq \ln C_{\kappa} \quad (2.7)$$

$$= \ln \kappa - \left[ \frac{1}{2\kappa} + \frac{1}{2\kappa^2} + \frac{7}{12\kappa^3} + \frac{3}{4\kappa^4} + \frac{31}{30\kappa^5} + \frac{3}{2\kappa^6} + O\left(\frac{1}{\kappa^7}\right) \right] \quad (2.8)$$

where we have carried out a large- $\kappa$  expansion.

On the other hand, lower bounds on  $N_{ST}(G)$  are more difficult to obtain. We have established an inequality which we state as a theorem.

**Theorem 2.1.** *Let  $G$  be a connected graph and let  $i$  and  $j$  be two non-adjacent vertices of  $G$  with degrees  $\kappa_i$  and  $\kappa_j$ , respectively. Let  $H$  be a graph obtained from  $G$  by adding an edge  $e_{ij}$  connecting  $i$  and  $j$ . Then we have*

$$N_{ST}(H) > \left( \frac{\kappa + 1}{\kappa} \right) N_{ST}(G) \quad (2.9)$$

where  $\kappa = \min\{\kappa_i, \kappa_j\}$ .

**Proof.** Since sites  $i$  and  $j$  are connected in  $G$ , the adding of the edge  $e_{ij}$  to any spanning tree  $T(G)$  on  $G$  forms a closed circuit on  $H$ . The closed circuit contains in addition to  $e_{ij}$  another edge  $\ell$  incident at site  $j$ . The deletion of  $\ell$  then breaks the circuit, resulting in a spanning tree configuration  $T(H)$  on  $H$ . However, the spanning tree  $T(H)$  thus constructed is not necessarily unique; the same  $T(H)$  may result from  $m$  different  $T(G)$ . Since each  $T(G)$  is also a spanning tree of  $H$ , it follows that for the  $m$  spanning trees  $T(G)$  there exist  $m + 1$  distinct spanning trees  $T(H)$ . A moment's reflection shows that we always have  $m \leq \kappa$ , with  $m = \kappa$  arising when there is a single edge incident at site  $j$  in  $T(G)$  and a single edge  $e_{ij}$  at  $j$  in  $T(H)$ . Since we have  $(\kappa_j + 1)/\kappa_j \geq (\kappa_i + 1)/\kappa_i$  for  $\kappa_i \geq \kappa_j$ , the theorem follows as a consequence.  $\square$

**Corollary 2.1.** *Let  $G$  be a  $\kappa$ -regular graph, and let  $G'$  be a graph derived from  $G$  by adding  $M$  edges one at a time such that each added edge terminates in at least one vertex whose degree is  $\kappa$ . Then*

$$N_{ST}(\mathcal{L}') > \left( \frac{\kappa + 1}{\kappa} \right)^M N_{ST}(\mathcal{L}). \quad (2.10)$$

**Remark.** Corollary 2.1 is proved by applying theorem 2.1  $M$  times. Furthermore, for lattices  $z_{\mathcal{L}}$  and  $z_{\mathcal{L}'}$ , and  $M = \alpha n$  where  $\alpha$  is a constant, equation (2.10) implies the bound

$$z_{\mathcal{L}'} > z_{\mathcal{L}} + \alpha \ln \left( \frac{\kappa + 1}{\kappa} \right). \quad (2.11)$$

For example, by adding edges one at a time one can convert the honeycomb lattice first to the square and then to the triangular lattice. Corollary 2.1 then implies the inequalities

$$z_{sq} > z_{hc} + \frac{1}{2} \ln \left( \frac{4}{3} \right) \quad (2.12)$$

and

$$z_{tri} > z_{sq} + \ln \left( \frac{5}{4} \right). \quad (2.13)$$

Combining these bounds with the relation  $z_{tri} = 2z_{hc}$  from (1.23), we obtain the lower bounds

$$z_{sq} > \ln \left( \frac{5}{3} \right) = 0.510\,8256\dots \quad (2.14)$$

$$z_{hc} > \ln \left( \frac{5}{2\sqrt{3}} \right) = 0.366\,9845\dots \quad (2.15)$$

$$z_{tri} > 2 \ln \left( \frac{5}{2\sqrt{3}} \right) = 0.733\,9691\dots \quad (2.16)$$

Note that these bounds have been deduced without actually carrying out explicit calculations. For comparison, the exact values are [6, 7] (see also section 4 below)

$$z_{sq} = \frac{4}{\pi} \left[ 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \dots \right] = 1.166\,2436\dots \quad (2.17)$$

and [7]

$$z_{hc} = \frac{3\sqrt{3}}{2\pi} \left[ 1 - \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{11^2} + \frac{1}{13^2} - \dots \right] = 0.807\,6649\dots \quad (2.18)$$

$$z_{tri} = 2z_{hc} = 1.615\,3297\dots \quad (2.19)$$

To measure the effectiveness of these lower bounds, let  $R_{\{G\}}$  denote the ratio of the exact  $z_{\{G\}}$  to the bounds (2.14)–(2.16). We have

$$R_{sq} \simeq 2.28 \quad R_{hc} = R_{tri} \simeq 2.20 \quad (2.20)$$

indicating that the bounds are very generous.

It is also of interest to work out the implications of the lower bound (2.11) for strips of regular lattices. For example, for  $2 \times \infty$  square [13, 14] and triangular [15] strips one has the results [13–15]

$$\begin{aligned} z_{sq(2 \times \infty)}^{free} &= \frac{1}{2} \ln(3 + \sqrt{2}) = 0.658\,4789\dots \\ z_{tri(2 \times \infty)}^{free} &= \frac{1}{2} \ln\left(\frac{1}{2}(7 + 3\sqrt{5})\right) = 0.962\,4236\dots \end{aligned} \quad (2.21)$$

where the superscript denotes free boundary conditions in the transverse direction. This is to be compared with (6.6.1) and (6.6.2) below for periodic boundary conditions in the transverse direction for which we have  $z_{sq(2 \times \infty)} = 0.881\,3735\dots$  and  $z_{tri(2 \times \infty)} = 1.386\,2943\dots$ . It is easily checked that these numbers obey (2.13). Furthermore, the inequality (2.13) now implies

$$\begin{aligned} z_{tri(2 \times \infty)}^{free} &> z_{sq(2 \times \infty)}^{free} + \frac{1}{2} \ln\left(\frac{4}{3}\right) = 0.802\,3199\dots \\ z_{tri(2 \times \infty)} &> z_{sq(2 \times \infty)} + \frac{1}{2} \ln\left(\frac{4}{3}\right) = 1.025\,2145\dots \end{aligned} \quad (2.22)$$

so that the ratios of the exact values to the lower bounds (2.22) are

$$R_{tri(2 \times \infty)}^{free} \simeq 1.20 \quad R_{tri(2 \times \infty)} \simeq 1.35. \quad (2.23)$$

Another example is the  $d$ -dimensional hypercubic lattice  $\mathcal{L}_d$  with  $\kappa = 2d$  for which corollary 2.1 implies†

$$z_{\mathcal{L}_{d+1}} > z_{\mathcal{L}_d} + \ln[1 + (2d)^{-1}]. \quad (2.24)$$

Applying this lower bound to  $z_{sq}$  using  $z_{line} = 0$ , we obtain  $z_{sq} > \ln(\frac{3}{2})$ , which is not as strong as the bound (2.14). For the  $d = 3$  simple cubic lattice, we have

$$z_{sc} > z_{sq} + \ln(\frac{5}{4}) = 1.389\,3871\dots \quad (2.25)$$

so that

$$R_{sc} \simeq 1.20. \quad (2.26)$$

Similar lower bounds can be deduced for hypercubic lattices with  $d \geq 4$ .

### 3. Families of graphs with exponential growth for $N_{ST}(G)$

In this section we prove a result concerning the class of families of graphs for which  $N_{ST}(G)$  has the exponential asymptotic behaviour (1.8).

A family of graphs is recursive if it can be built up by sequential additions of a given subgraph. As an example, consider a strip of width  $N_2$  and length  $N_1 = m$  of some lattice such as a square lattice; this can be built up by starting with a column of  $N_2 - 1$  squares and sequentially adding columns to elongate the strip in the  $N_1$  direction. A higher-dimensional example is a rectangular tube of a lattice such as a simple cubic lattice with transverse size  $N_2 \times N_3$  and length  $N_1 = m$ ; this can be built up by starting with a single  $N_2 \times N_3$  section and sequentially adding  $m$  transverse sections and connecting them in an obvious manner to elongate the tube. We will need the following result from [14].

**Lemma 3.1.** *Let  $G_m$  be a recursive graph of length  $m$  subunits as described in the above. Then the Tutte polynomial has the form*

$$T(G_m, x, y) = \sum_{j=1}^{N_a} c_{G,j}(x, y) (a_{G,j}(x, y))^m \quad (3.1)$$

where the explicit forms of  $a_{G,j}(x, y)$  and  $c_{G,j}(x, y)$  depend on the type of graphs  $G_m$ .

The proof of lemma 3.1 can be found in [14]. (See (2.18) and (8.15) of [14], where  $a_{G,j}$  was denoted by  $\lambda_{G,j}$ .) Note that the class of recursive graphs is more general than the class of regular lattices.

As discussed in [14], as  $m \rightarrow \infty$  for a given  $(x, y)$ , the term  $a_{G,j}$  with the maximal magnitude will dominate the right-hand side of (3.1), provided that the corresponding coefficient  $c_{G,j}(x, y)$  does not vanish. We denote this term as  $a_{G,j,max}$ . Using the relation between the Potts partition function and the Tutte polynomial (1.13), together with the definition of the (reduced) Potts model free energy

$$f(\{G\}, q, v) = \lim_{n \rightarrow \infty} n^{-1} \ln Z(G, q, v) \quad (3.2)$$

† Here we add a technical remark. As the inequality (2.24) is deduced by constructing  $\mathcal{L}_{d+1}$  via adding edges to connect  $n^{1/d}$  copies of  $\mathcal{L}_d$ , a technical problem arises if the copies are disjoint to begin with for which  $N_{ST} = 0$  by definition. The difficulty is resolved if one starts instead from copies of  $\mathcal{L}_d$  which are connected by a sequence of single edges.



one observes that a non-analyticity in  $f$  occurs when, as one changes  $(x, y)$  or equivalently  $(q, v)$ , there is a crossover of the dominant term  $a_{G,j}$ . These changes determine the regions of analyticity (phases) of the free energy. We next proceed to our theorem.

**Theorem 3.1.** *Let  $G_m$  denote a recursive graph of a lattice of length  $m$  in one spatial dimension with fixed  $(d - 1)$ -dimensional ‘transverse’ section of the size  $N_2 \times N_3 \times \cdots \times N_d$ . For  $N_2, \dots, N_d \geq 2$ , the number of spanning trees  $N_{ST}(G_m)$  grows exponentially with  $n$  as in (1.8), thereby defining a non-zero finite constant  $z_{\{G\}}$ .*

**Proof.** We shall carry out the proof for the case  $d = 2$ ; the generalization to  $d \geq 3$  is straightforward. The strategy of the proof is to use the structural result (3.1) in lemma 3.1 above. It is convenient to use some results from the Potts–Tutte correspondence (1.13). From (1.12) and (1.14) it follows that  $y = 1$  corresponds, in terms of the correspondence (1.13), to the infinite-temperature point  $K = v = 0$ . From the basic property that a spin model such as the Potts model is analytic at  $K = 0$ , i.e. has a Taylor series expansion in  $K$  (or  $v$ ) with a finite radius of convergence, it follows that, in the neighbourhood of  $y = 1$  for a given  $x$ , a single term

$$a_{G,max} = a_{G,R_1} \quad (3.3)$$

will dominate, where  $R_1$  denotes the region in the  $(x, y)$ -space corresponding to the paramagnetic phase for a given  $q$  in the  $(q, v)$ -space. It was shown in [14] that the corresponding coefficient in  $T(G, x, y)$  is non-zero. Further, one knows that  $a_{G,max}(1, 1) > 1$  unless  $G$  is the tree or circuit graph or obvious modifications thereof, a fact which can be easily proved by assuming the contrary and deducing that  $N_{ST}$  violates a lower bound on the number of spanning trees for strip graphs with width  $N_2 \geq 2$ . From (1.6) and (1.9), it follows that  $N_{ST}(G_m)$  has the exponential asymptotic growth as in (1.8), so that there is a finite non-zero constant  $z_{\{G\}}$ . The method of the proof evidently provides a constructive way to calculate this quantity in terms of (3.1), or

$$z_{\{G\}} = t^{-1} \ln[a_{G,j,max}(1, 1)] \quad (3.4)$$

where  $t$  is the number of vertices in a transverse section. It is straightforward to generalize these results to the case of a  $d$ -dimensional tube graph with fixed  $(d - 1)$ -dimensional transverse section. This completes the proof.  $\square$

To place this result in perspective, we give some examples of families of graphs for which the asymptotic behaviour (1.8) does not hold. For the tree graph  $T_n$  and circuit graph  $C_n$ , the Tutte polynomials are  $T(T_n, x, y) = x^{n-1}$  and  $T(C_n, x, y) = y + \sum_{s=1}^{n-1} x^s$  so that  $N_{ST}(T_n) = 1$ , independent of  $n$ , and  $N_{ST}(C_n) = n$ . In both cases  $z_{\{T\}} = z_{\{C\}} = 0$ . It is for this reason that we restricted the width of the strips to  $N_2, N_3, \dots \geq 2$  in theorem 3.1. An example of a family for which  $N_{ST}$  grows more rapidly than an exponential is the complete graph  $K_n$ , a graph with the property that each vertex is adjacent to every other vertex. In this case, one has  $N_{ST}(K_n) = n^{n-2}$  [2]. Note that  $K_n$  is not a recursive graph. We next proceed to the case of the bulk limit of a lattice.

**Theorem 3.2.** *Let  $\mathcal{L}$  denote the bulk limit of a regular  $d$ -dimensional lattice of size  $N_1 \times \cdots \times N_d$ , with  $N_j \rightarrow \infty$ ,  $j = 1, \dots, d$  such that the ratios  $\lim_{n \rightarrow \infty} N_i/N_j$  remain non-zero and finite. Then the number of spanning trees on this lattice grows exponentially with  $n$  as in (1.8), thereby defining a non-zero finite constant  $z_{\mathcal{L}}$ .*

**Proof.** A sketch of the proof is given here. The idea is to use theorem 3.1 and observe that the property of exponential growth of  $N_{ST}$  as  $N_1 \rightarrow \infty$  for fixed  $N_2$  is independent of  $N_2 \geq 2$ . Furthermore, as discussed in the proof of theorem 3.1, given the relation (1.6), it follows that  $N_{ST}$  and  $z$  are, in statistical mechanics terminology, determined by  $Z(G, q, v)$  and the free energy  $f$  at the disorder point  $v = K = 0$ . Hence, we can take the limit  $N_2 \rightarrow \infty$ , and since the exponential growth of  $N_{ST}$  holds uniformly in  $N_2$ , it also holds in this limit. This establishes the result for  $d = 2$ . It is straightforward to extend the proof to  $d \geq 3$  by using theorem 3.1 starting with a tube of the  $d$ -dimensional lattice with fixed  $(d - 1)$ -dimensional transverse cross section and one longitudinal direction that goes to infinity, and using again the fact that the exponential growth of  $N_{ST}$  holds uniformly as one increases the  $(d - 1)$  transverse dimensions of the tube.  $\square$

We remark that theorem 3.2 can also be established directly using the explicit expression (4.12) of  $N_{ST}(\mathcal{L})$  obtained in the next section. Another remark is that it is important that  $z$  is a disorder quantity as defined above, i.e. that it corresponds to the infinite-temperature limit of the Potts model. Because of this, one obtains the same asymptotic behaviour for  $z$  on a sequence of infinite-length tube graphs of progressively larger and larger transverse cross sections as one does by taking the limit of all  $N_j \rightarrow \infty$  with the ratios  $N_i/N_j$  fixed. In contrast, this would not be the case in dealing with a quantity connected with a divergent correlation length. For example, the free energy of the ferromagnetic Potts model on infinite-length, finite-width strips has a zero-temperature critical point at  $v \rightarrow \infty$  for any finite width, but has a quite different analytic structure if one takes both  $N_1$  and  $N_2$  to infinity with  $N_2/N_1$  fixed, namely, a non-analyticity (phase transition) at a finite temperature  $v > 1$ .

#### 4. Formulation for general lattices

The formulation of enumerating spanning trees for general lattices is given in this section. Consider a lattice  $\mathcal{L}$  of  $n$  sites in  $d$  spatial dimensions. We shall use (1.5) to evaluate  $N_{ST}$  and for simplicity assume periodic boundary conditions. Formulations for other boundary conditions can be similarly worked out (see, for example, [11]).

To write down the Laplacian matrix  $Q$  in a form suitable for computing its eigenvalues, we make use of the fact that any lattice in  $d$  dimensions is decomposable into a hypercubic array of  $N_1 \times N_2 \times \dots \times N_d$  unit cells, each containing  $v$  sites so that we have  $n = vN_1N_2 \dots N_d$ <sup>†</sup>. Specify the cells by the coordinate  $\mathbf{n} = \{n_1, n_2, \dots, n_d\}$ , where  $n_i = 0, 1, 2, \dots, N_i - 1$ , and number the sites in a cell  $1, 2, \dots, v$ . Let  $a(\mathbf{n}, \mathbf{n}')$  be the  $v \times v$  cell vertex adjacency matrix describing the connectivity between the vertices of the unit cells  $\mathbf{n}$  and  $\mathbf{n}'$ . Namely,

$$a_{ij}(\mathbf{n}, \mathbf{n}') = \begin{cases} 1 & \text{if site } i \text{ in cell } \mathbf{n} \text{ and site } j \text{ in cell } \mathbf{n}' \text{ are adjacent} \\ 0 & \text{otherwise.} \end{cases} \quad (4.1)$$

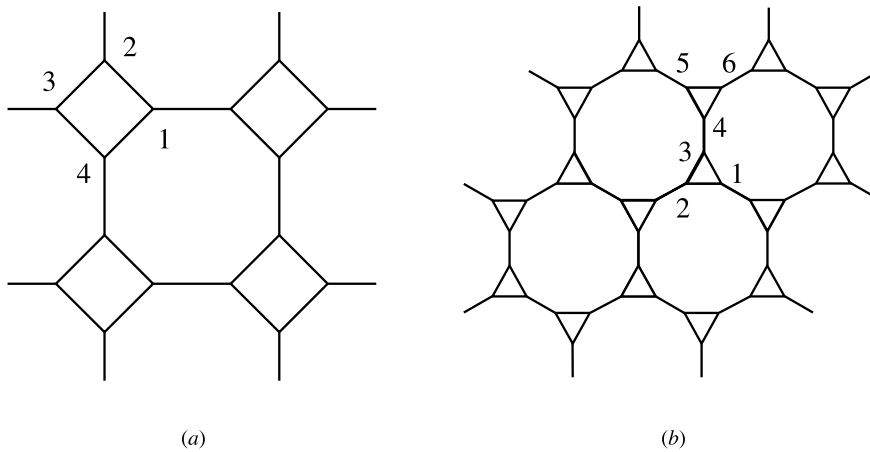
Under the assumption of periodic boundary conditions, we have the translational symmetry

$$a(\mathbf{n}, \mathbf{n}') = a(\mathbf{n} - \mathbf{n}') \quad (4.2)$$

and we can therefore write  $a(\mathbf{n}) = a(n_1, n_2, \dots, n_d)$ .

The general formulation is best illustrated by considering an example. Here we consider the 4–8–8 (bathroom-tile) lattice shown in figure 1(a). This is a regular lattice which has the coordination number 3, and the unit cells are the obliquely oriented squares, with  $v = 4$ . The

<sup>†</sup> If  $N_j = 2$  for some  $j$ , then the two sites in the  $j$ th direction are connected by double edges. However, this does not affect any of the ensuing discussions.



**Figure 1.** (a) The 4–8–8 (bathroom tile) lattice. (b) The 3–12–12 lattice. Sites within a unit cell are labelled as shown.

explicit forms for the  $a(n_1, n_2)$  matrices depend on one’s convention for labelling the vertices within a unit cell; we choose the labelling shown in figure 1(a). Then one has

$$\begin{aligned}
 a(0, 0) &= \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \\
 a(1, 0) = a^T(-1, 0) &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
 a(0, 1) = a^T(0, -1) &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
 \end{aligned} \tag{4.3}$$

where  $a^T$  is the transpose of  $a$ . Then the Laplacian matrix  $Q$  assumes the form  $Q = [3I_4 - a(0, 0)] \otimes I_{N_1} \otimes I_{N_2} - a(1, 0) \otimes R_{N_1} \otimes I_{N_2} + a(-1, 0) \otimes R_{N_1}^T \otimes I_{N_2}$

$$-a(0, 1) \otimes I_{N_1} \otimes R_{N_2} - a(0, -1) \otimes I_{N_1} \otimes R_{N_2}^T \tag{4.4}$$

where  $I_N$  is the  $N \times N$  identity matrix and  $R_N$  is the  $N \times N$  matrix

$$R_N = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{4.5}$$

To determine the eigenvalues of  $Q$ , we make use of the fact that  $R_N$  is diagonalized by the similarity transformation  $S_N R_N S_N^{-1}$  generated by the matrix  $S_N$  with elements

$$(S_N)_{nm} = (S_N^{-1})_{mn}^* = N^{-1/2} e^{i2\pi mn/N} \quad m, n = 0, 1, \dots, N - 1 \tag{4.6}$$

where  $*$  denotes the complex conjugate, yielding the eigenvalues

$$\lambda_n = e^{i2\pi n/N} \quad n = 0, 1, \dots, N - 1. \tag{4.7}$$

It follows that the similarity transformation generated by

$$I_\nu \otimes S_{N_1} \otimes S_{N_2} \tag{4.8}$$

diagonalizes  $Q$  in the  $N_1$  and  $N_2$  subspaces. Then, using the fact that a determinant is equal to the product of its eigenvalues, we obtain from (1.5) the expression

$$N_{ST}(\mathcal{L}_{4-8-8}) = \left( \frac{\Lambda}{4N_1N_2} \right) \prod_{k_1=0}^{N_1-1} \prod_{k_2=0}^{N_2-1} \det \left| M \left( \frac{2\pi k_1}{N_1}, \frac{2\pi k_2}{N_2} \right) \right| \quad (k_1, k_2) \neq (0, 0) \tag{4.9}$$

where  $\Lambda = \lambda_1\lambda_2\lambda_3 = 64$ , and  $\lambda_1 = \lambda_2 = \lambda_3 = 4$  are the non-zero eigenvalues of  $M(0, 0)$ . Here,  $M$  is a  $4 \times 4$  matrix and the notation  $\det |M|$  denotes the determinant of  $M$ . Explicitly, we have

$$\begin{aligned} M(\theta_1, \theta_2) &= 3I_4 - a(0, 0) - a(1, 0) e^{i\theta_1} - a(-1, 0) e^{-i\theta_1} - a(0, 1) e^{i\theta_2} - a(0, -1) e^{-i\theta_2} \\ &= \begin{pmatrix} 3 & -1 & -e^{i\theta_1} & -1 \\ -1 & 3 & -1 & -e^{i\theta_2} \\ -e^{-i\theta_1} & -1 & 3 & -1 \\ -1 & -e^{-i\theta_2} & -1 & 3 \end{pmatrix} \end{aligned}$$

and hence

$$\det |M(\theta_1, \theta_2)| = 4[7 - 3(\cos \theta_1 + \cos \theta_2) - \cos \theta_1 \cos \theta_2]. \tag{4.10}$$

Note that a change in labelling conventions would either interchange or negate  $\theta_1$  and/or  $\theta_2$ . These have no effect on the final expression since  $\det |M|$  is invariant under these changes. From (4.10) we have

$$\begin{aligned} z(\mathcal{L}_{4-8-8}) &= \frac{1}{2} \ln 2 + \frac{1}{4} \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_2}{2\pi} \ln [7 - 3(\cos \theta_1 + \cos \theta_2) - \cos \theta_1 \cos \theta_2] \\ &= \frac{1}{4} \ln 2 + \frac{1}{4\pi} \int_0^\pi d\theta \ln [7 - 3 \cos \theta + 4 \sin(\theta/2) \sqrt{5 - \cos \theta}] \\ &= 0.786\,684(1). \end{aligned} \tag{4.11}$$

The last two lines are obtained after carrying out one integration followed by a numerical integration of the remaining integral.

The consideration of general  $\mathcal{L}$  now proceeds in a similar fashion. In place of (4.9), one obtains

$$N_{ST}(\mathcal{L}) = \left( \frac{\Lambda}{\nu N_1 \dots N_d} \right) \prod_{k_1=0}^{N_1-1} \dots \prod_{k_d=0}^{N_d-1} D \left( \frac{2\pi k_1}{N_1}, \dots, \frac{2\pi k_d}{N_d} \right) \quad (\mathbf{k} \neq \mathbf{0}) \tag{4.12}$$

where  $\mathbf{k} = (k_1, k_2, \dots, k_d)$ ,  $\mathbf{0} = (0, 0, \dots, 0)$ ,

$$D(\theta_1, \dots, \theta_d) = \det |M(\theta_1, \dots, \theta_d)| \tag{4.13}$$

$$\Lambda = \begin{cases} 1 & \nu = 1 \\ \lambda_1 \dots \lambda_{\nu-1} & \nu > 1 \end{cases} \tag{4.14}$$

and  $\lambda_i$ 's are the  $\nu - 1$  non-zero eigenvalues of the matrix  $M(0, \dots, 0)$ . Here,  $M$  is a  $\nu \times \nu$  matrix defined by

$$M(\theta_1, \dots, \theta_d) = \Delta_\nu - \sum_n a(\mathbf{n}) e^{i\mathbf{n} \cdot \Theta} \tag{4.15}$$

where  $\Delta_\nu$  is the degree matrix (1.2) for a unit cell, and  $\Theta = (\theta_1, \dots, \theta_d)$ . Note that the determinant of  $M$  is always real since the matrix  $M$  is Hermitian. This leads to the result

$$z_{\mathcal{L}} \equiv z(\mathcal{L}) = \frac{1}{\nu} \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \dots \int_{-\pi}^{\pi} \frac{d\theta_d}{2\pi} \ln D(\theta_1, \dots, \theta_d). \tag{4.16}$$

Equations (4.12) and (4.16) are our main results for general regular lattices, and (4.12) is suitable for enumerating  $N_{ST}$  for lattices of finite sizes.

It is also of interest to consider the case in which the size of the lattice is finite in  $\ell < d$  dimensions and goes to infinity in the remaining  $d - \ell$  dimensions. For example, if one sets  $\ell = 1$ , then as  $N_1$  increases from 1 to infinity, the resultant sequence of values of  $z$  can be regarded as a sort of 'interpolation' between the infinite  $(d - 1)$ -dimensional and infinite  $d$ -dimensional lattices. Without loss of generality we let  $L_1, \dots, L_\ell$  be finite. Then from (4.12) we have

$$z(\mathcal{L}(N_1 \times \dots \times N_\ell \times \infty \times \dots \times \infty)) = \frac{1}{\nu N_1 \dots N_\ell} \times \sum_{k_1=0}^{N_1-1} \dots \sum_{k_\ell=0}^{N_\ell-1} \int_{-\pi}^{\pi} \frac{d\theta_{\ell+1}}{2\pi} \dots \int_{-\pi}^{\pi} \frac{d\theta_d}{2\pi} D\left(\frac{2\pi k_1}{N_1}, \dots, \frac{2\pi k_\ell}{N_\ell}, \theta_{\ell+1}, \dots, \theta_d\right). \tag{4.17}$$

This shows clearly the result of theorem 3.1, that  $N_{ST}$  grows exponentially like (1.8). For  $d = 2, \ell = 1$ , i.e. the case of  $N \times \infty$  strips of infinite length and finite width  $N_1 \equiv N$ , the integration can be carried out. In particular, for strips of square and triangular lattices, we obtain the expressions

$$z_{sq(N \times \infty)} = \frac{1}{N} \sum_{k=0}^{N-1} \ln[2 - \cos \omega_k + ((2 - \cos \omega_k)^2 - 1)^{1/2}] \tag{4.18}$$

$$z_{tri(N \times \infty)} = \frac{1}{N} \sum_{k=0}^{N-1} \ln[3 - \cos \omega_k + ((1 - \cos \omega_k)(7 - \cos \omega_k))^{1/2}] \tag{4.19}$$

where  $\omega_k = 2\pi k/N$ . The summations (4.18) and (4.19) can be explicitly carried out for a given  $N$ . We give the results in section 6.6 below.

### 5. Lattices in $d \geq 3$ dimensions

The formulation of the preceding section is now specialized to specific lattices in  $d \geq 3$  dimensions.

#### 5.1. $d$ -dimensional hypercubic lattices

Consider first the  $d$ -dimensional hypercubic lattice  $\mathcal{L}_d$  consisting of  $n = N_1 N_2 \dots N_d$  sites. Here,  $\mathcal{L}_2$  and  $\mathcal{L}_3$  are the square and simple cubic lattices. We have  $\nu = 1, \Delta_1 = \kappa = 2d$ , and the adjacency matrices

$$a(\mathbf{n}) = \begin{cases} 1 & \mathbf{n} = (\pm 1, 0, \dots, 0), (0, \pm 1, \dots, 0), \dots, (0, 0, \dots, \pm 1) \\ = 0 & \text{otherwise.} \end{cases} \tag{5.1.1}$$

Using the general expressions (4.12) and (4.16), one obtains immediately

$$N_{ST}(\mathcal{L}_d) = \frac{2^{n-1}}{n} \prod_{k_1=0}^{N_1-1} \cdots \prod_{k_d=0}^{N_d-1} \left[ d - \left( \cos\left(\frac{2k_1\pi}{N_1}\right) + \cdots + \cos\left(\frac{2k_d\pi}{N_d}\right) \right) \right] \quad (\mathbf{k} \neq \mathbf{0}). \quad (5.1.2)$$

From (5.1.2) we derive

$$z(\mathcal{L}_d) = \ln(2d) + \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \cdots \int_{-\pi}^{\pi} \frac{d\theta_d}{2\pi} \ln \left[ 1 - \frac{1}{d} (\cos \theta_1 + \cdots + \cos \theta_d) \right]. \quad (5.1.3)$$

The expression (5.1.2) is the same as that reported in [11].

As in the case of the 4–8–8 lattice, one of the integrations can be carried out analytically and the remaining  $(d-1)$ -fold integral done numerically. For  $d=2$ , the quantity  $z(\mathcal{L}_2)$  can be evaluated exactly [6, 7], leading to the value given by (2.17) which we include in (5.1.5) below. We have carried out the numerical integrations for  $d=3, 4$  and also present the results in (5.1.5). For higher values of  $d$  the evaluation of the integral becomes less accurate; however, one can expand the logarithm in (5.1.3) and carry out the integrations term by term. This leads to the large- $d$  expression

$$z(\mathcal{L}_d) = \ln(2d) - \left[ \frac{1}{4d} + \frac{3}{16d^2} + \frac{7}{32d^3} + \frac{45}{128d^4} + \frac{269}{384d^5} + \frac{805}{512d^6} + O\left(\frac{1}{d^7}\right) \right]. \quad (5.1.4)$$

Note that the expansion (5.1.4) agrees with the corresponding large- $\kappa$  expansion (2.8) of the McKay–Fan–Yau upper bound to the order of  $d^{-1}$ , after taking  $\kappa = 2d$ . However,  $1/d$  expansions are generally expected to be asymptotic in view of the slow growth of the coefficients (see, for example, [24]). If one truncates the  $1/d$  series to a given order and lets  $d \rightarrow \infty$ , the approximation to  $z(\mathcal{L}_d)$  becomes progressively more accurate. However, for a fixed  $d$ , one does not necessarily obtain a more accurate approximation by including more terms in the calculation. In practice, however, we found that, for  $d=4, 5$  and  $6$ , the series evaluated to  $O(d^{-4})$  and  $O(d^{-5})$  gives essentially the same values, which we listed in (5.1.5), with an accuracy of  $10^{-4}$ . Combining our results, we have

$$\begin{aligned} z_{sq} &= z(\mathcal{L}_2) = 1.166\,2436\dots \\ z(\mathcal{L}_3) &= 1.674\,1481(1) && \text{(numerical evaluation)} \\ z(\mathcal{L}_4) &= 2.0000(5) && \text{(numerical evaluation)} \\ z(\mathcal{L}_5) &= 2.243 && \text{(series expansion)} \\ z(\mathcal{L}_6) &= 2.437 && \text{(series expansion)} \\ z(\mathcal{L}_d) &\rightarrow \ln(2d) && d \rightarrow \infty. \end{aligned} \quad (5.1.5)$$

Our numerical result suggests that  $z(\mathcal{L}_4)$  may be exactly equal to 2. It is readily verified that values of  $z(\mathcal{L}_d)$  in (5.1.5) are consistent with the inequality (2.24).

## 5.2. $d$ -dimensional body-centred cubic lattice

For the usual three-dimensional body-centred cubic (bcc) lattice, a unit cell contains  $\nu = 2$  vertices located at  $(0, 0, 0)$  and  $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$  numbered 1 and 2, respectively. Then one has

$\Delta_2 = 8I_2$  and the adjacency matrices

$$\begin{aligned}
 a(0, 0, 0) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
 a(1, 0, 0) &= a(0, 1, 0) = a(0, 0, 1) = a(1, 1, 0) \\
 &= a(1, 0, 1) = a(0, 1, 1) = a(1, 1, 1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
 a(-1, 0, 0) &= a(0, -1, 0) = a(0, 0, -1) = a(-1, -1, 0) \\
 &= a(-1, 0, -1) = a(0, -1, -1) = a(-1, -1, -1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
 \end{aligned} \tag{5.2.1}$$

leading to the matrix

$$M(\theta_1, \theta_2) = \begin{pmatrix} 8 & -v_1 v_2 v_3 \\ -(v_1 v_2 v_3)^* & 8 \end{pmatrix} \tag{5.2.2}$$

where  $v_j = 1 + e^{i\theta_j}$ ,  $j = 1, 2, 3$ . The evaluation of the determinant yields

$$D(\theta_1, \theta_2) = 64[1 - \cos^2(\theta_1/2) \cos^2(\theta_2/2) \cos^2(\theta_3/2)]. \tag{5.2.3}$$

This leads to

$$\begin{aligned}
 N_{ST}(\mathcal{L}_{bcc}) &= \left(\frac{\Lambda}{n}\right) \prod_{k_1=0}^{N_1-1} \prod_{k_2=0}^{N_2-1} \prod_{k_3=0}^{N_3-1} \left[64 - 64 \cos^2\left(\frac{k_1\pi}{N_1}\right) \cos^2\left(\frac{k_2\pi}{N_2}\right) \cos^2\left(\frac{k_3\pi}{N_3}\right)\right] \\
 &\quad (k_1, k_2, k_3) \neq (0, 0, 0)
 \end{aligned} \tag{5.2.4}$$

where  $n = 2N_1N_2N_3$ , and  $\Lambda = 16$  is the non-zero eigenvalue of  $M(0, 0)$ . The expression (5.2.4) enumerates  $N_{ST}$  for finite bcc lattices. Using (1.9), we obtain

$$\begin{aligned}
 z(\mathcal{L}_{bcc}) &= 3 \ln 2 + \frac{1}{2} \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_2}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_3}{2\pi} \ln [1 - \cos^2(\frac{1}{2}\theta_1) \cos^2(\frac{1}{2}\theta_2) \cos^2(\frac{1}{2}\theta_3)] \\
 &= 3 \ln 2 + \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_2}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_3}{2\pi} \ln(1 - \cos \theta_1 \cos \theta_2 \cos \theta_3).
 \end{aligned} \tag{5.2.5}$$

These results can be generalized to the  $d$ -dimensional body-centred cubic lattice, which we shall denote as  $bcc(d)$ . This lattice has coordination number  $\kappa = 2^d$ . For finite lattices we obtain

$$N_{ST}(\mathcal{L}_{bcc(d)}) = \left(\frac{\Lambda}{2N_1 \cdots N_d}\right) \prod_{k_1=0}^{N_1-1} \cdots \prod_{k_d=0}^{N_d-1} \left[2^{2d} - 2^{2d} \prod_{j=1}^d \cos^2\left(\frac{k_j\pi}{N_j}\right)\right] \quad (\mathbf{k} \neq \mathbf{0}) \tag{5.2.6}$$

where  $\Lambda = 2^{d+1}$ . Taking the bulk limit, we obtain

$$\begin{aligned}
 z(\mathcal{L}_{bcc(d)}) &= d \ln 2 + \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \cdots \int_{-\pi}^{\pi} \frac{d\theta_d}{2\pi} \ln(1 - (\cos \theta_1) \cdots (\cos \theta_d)) \\
 &= d \ln 2 - \frac{1}{2} \sum_{\ell=1}^{\infty} \frac{1}{\ell} \left(\frac{(2\ell)!}{2^{2\ell} (\ell!)^2}\right)^d
 \end{aligned} \tag{5.2.7}$$

where we have expanded the logarithm and carried out the integrations term by term.

For  $d = 2$ , the body-centred cubic lattice  $bcc(2)$  is just the square lattice, and it is readily seen that the two integrals in (5.1.3) and (5.2.7) are equal for  $d = 2$ . Interestingly, this also

establishes the equality of the two series in (2.17) and (5.2.7) at  $d = 2$ . We have further evaluated  $z(\mathcal{L}_{bcc(d)})$  for  $d \geq 3$  using both expressions in (5.2.7), and found that the series converges slowly, with good agreement between the two reached only after evaluating the series to 100–200 terms. For  $d = 3$  and 4 the results are

$$\begin{aligned} z(\mathcal{L}_{bcc}) &= 1.9902(1) & d &= 3 \\ z(\mathcal{L}_{bcc(4)}) &= 2.7323(1) & d &= 4. \end{aligned} \tag{5.2.8}$$

### 5.3. Face-centred cubic lattice

For the face-centred cubic (fcc) lattice, a unit cell contains  $\nu = 4$  vertices located at  $(0, 0, 0)$ ,  $(0, \frac{1}{2}, \frac{1}{2})$ ,  $(\frac{1}{2}, 0, \frac{1}{2})$ ,  $(\frac{1}{2}, \frac{1}{2}, 0)$  numbered 1, 2, 3, 4, respectively. One has  $\Delta_4 = (12)I_4$  and the adjacency matrices,

$$\begin{aligned} a(0, 0, 0) &= \begin{pmatrix} 0 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{pmatrix} \\ a(1, 0, 0) &= a^T(-1, 0, 0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \\ a(0, 1, 0) &= a^T(0, -1, 0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \\ a(0, 0, 1) &= a^T(0, 0, -1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ a(1, 1, 0) &= a^T(-1, -1, 0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\ a(1, 0, 1) &= a^T(-1, 0, -1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ a(0, 1, 1) &= a^T(0, -1, -1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{5.3.1}$$



$$a(1, -1, 0) = a^T(-1, 1, 0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$a(1, 0, -1) = a^T(-1, 0, 1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$a(0, 1, -1) = a^T(0, -1, 1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

leading to the matrix

$$M(\theta_1, \theta_2, \theta_3) = \begin{pmatrix} 12 & -(v_2 v_3)^* & -(v_1 v_3)^* & -(v_1 v_2)^* \\ -v_2 v_3 & 12 & -v_1^* v_2 & -v_1^* v_3 \\ -v_1 v_3 & -v_1 v_2^* & 12 & -v_2^* v_3 \\ -v_1 v_2 & -v_1 v_3^* & -v_2 v_3^* & 12 \end{pmatrix} \quad (5.3.2)$$

where  $v_j = 1 + e^{i\theta_j}$ ,  $j = 1, 2, 3$ . The evaluation of the determinant yields

$$D(\theta_1, \theta_2, \theta_3) = 12^4 F(\theta_1, \theta_2, \theta_3) \quad (5.3.3)$$

$$F(\theta_1, \theta_2, \theta_3) = 1 - \frac{2}{9}(c_1 + c_2 + c_3) - \frac{8}{27}c_1 c_2 c_3 - \frac{2}{81}c_1 c_2 c_3 (c_1 + c_2 + c_3) + \frac{1}{81}(c_1^2 + c_2^2 + c_3^2)$$

where  $c_i = \cos^2(\theta_i/2)$ . This gives

$$N_{ST}(\mathcal{L}_{fcc}) = \left(\frac{\Lambda}{n}\right) \prod_{k_1=0}^{N_1-1} \prod_{k_2=0}^{N_2-1} \prod_{k_3=0}^{N_3-1} D\left(\frac{2k_1\pi}{N_1}, \frac{2k_2\pi}{N_2}, \frac{2k_3\pi}{N_3}\right) \quad (k_1, k_2, k_3) \neq (0, 0, 0) \quad (5.3.4)$$

where  $\Lambda = 16^3$ ,  $n = 4N_1 N_2 N_3$ , so that

$$z_{fcc} = \ln(12) + \frac{1}{4} \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_2}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_3}{2\pi} \ln F(\theta_1, \theta_2, \theta_3). \quad (5.3.5)$$

The numerical evaluation of (5.3.5) yields the value

$$z_{fcc} = 2.3544(4). \quad (5.3.6)$$

## 6. Planar lattices

In this section the formulation of section 4 is applied to some other planar lattices (the square and 4–8–8 lattices have been considered in preceding sections).

### 6.1. Triangular lattice

The triangular lattice can be regarded as an  $N_1 \times N_2$  square net of sites with one additional diagonal edge added, in the same way, to every square of the net. In this picture we have  $\nu = 1$ ,  $a(\pm 1, 0) = a(0, \pm 1) = a(1, 1) = a(-1, -1) = 1$ , and

$$M(\theta_1, \theta_2) = 6 - (e^{i\theta_1} + e^{-i\theta_1} + e^{i\theta_2} + e^{-i\theta_2} + e^{i(\theta_1+\theta_2)} + e^{-i(\theta_1+\theta_2)}). \quad (6.1.1)$$

It follows that

$$\begin{aligned} z_{tri} &= \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_2}{2\pi} \ln[6 - 2(\cos \theta_1 + \cos \theta_2 + \cos(\theta_1 + \theta_2))] \\ &= \frac{3\sqrt{3}}{\pi} (1 - 5^{-2} + 7^{-2} - 11^{-2} + 13^{-2} - \dots) \\ &= 1.615\,329\,736\dots \end{aligned} \quad (6.1.2)$$

The result (6.1.2) was reported previously in [7] where it was derived via the connection to the Potts partition function and a mapping to a solvable vertex model. For a finite triangular lattice of  $N_1 \times N_2$  cells, the number of spanning trees  $N_{ST}$  is given by (4.12) with  $d = 2$ ,  $\Lambda = \nu = 1$ , an expression we shall not reproduce here.

### 6.2. Honeycomb lattice

It is instructive to derive  $z_{hc}$  using the general formulation. The honeycomb lattice is a square net of unit cells of  $\nu = 2$  sites. Consider the honeycomb lattice in the form of a ‘brick wall’ and regard the two sites connected by a vertical edges as forming a unit cell. Then one has

$$a(0, 0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad a(1, 0) = a(-1, 0) = a^T(0, 1) = a^T(0, -1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (6.2.1)$$

and

$$M(\theta_1, \theta_2) = \begin{pmatrix} 3 & -(1 + e^{i\theta_1} + e^{i\theta_2}) \\ -(1 + e^{-i\theta_1} + e^{-i\theta_2}) & 3 \end{pmatrix} \quad (6.2.2)$$

$$D(\theta_1, \theta_2) = 6 - 2(\cos \theta_1 + \cos \theta_2 + \cos(\theta_1 + \theta_2)).$$

This leads to the relation  $z_{hc} = z_{tri}/2$  given in (1.23). For finite honeycomb lattices the number of spanning trees is the same as that of its dual, the triangular lattice.

### 6.3. Kagomé and diced lattices

The kagomé lattice has the structure of a square net of unit cells each containing  $\nu = 3$  sites forming a triangle. Therefore, one has

$$a(0, 0) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad a(1, 0) = a^T(-1, 0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad (6.3.1)$$

$$a(0, 1) = a^T(0, -1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad a(1, 1) = a^T(-1, -1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and

$$M(\theta_1, \theta_2) = \begin{pmatrix} 4 & -(1 + e^{i\theta_2}) & -(1 + e^{-i\theta_1}) \\ -(1 + e^{-i\theta_2}) & 4 & -(1 + e^{-i(\theta_1 + \theta_2)}) \\ -(1 + e^{i\theta_1}) & -(1 + e^{i(\theta_1 + \theta_2)}) & 4 \end{pmatrix} \quad (6.3.2)$$

with

$$D(\theta_1, \theta_2) = 12[3 - (\cos \theta_1 + \cos \theta_2 + \cos(\theta_1 + \theta_2))]. \quad (6.3.3)$$

This yields the result

$$z_{kag} = (z_{tri} + \ln 6)/3. \quad (6.3.4)$$

For a finite kagomé lattice of  $N_1 \times N_2$  cells, the number of spanning trees  $N_{ST}$  is given by (4.12) with  $d = 2$ ,  $\Lambda = 6^2$ ,  $\nu = 3$ .

For the diced lattice, which is the dual of the kagomé lattice, from (1.17) and (1.23), we obtain

$$N_{ST}(\mathcal{L}_{diced}) = N_{ST}(\mathcal{L}_{kag}) \quad z(\mathcal{L}_{diced}) = z(\mathcal{L}_{kag}). \quad (6.3.5)$$

#### 6.4. 3–12–12 lattice

The 3–12–12 lattice is the lattice shown in figure 1(b) which has the structure of a square net with unit cells each containing  $\nu = 6$  sites. Labelling the six sites of a unit cell as shown, one has

$$M(\theta_1, \theta_2) = \begin{pmatrix} 3 & -1 & -1 & 0 & -e^{i\theta_1} & 0 \\ -1 & 3 & -1 & 0 & 0 & -e^{-i\theta_2} \\ -1 & -1 & 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & -1 \\ -e^{-i\theta_1} & 0 & 0 & -1 & 3 & -1 \\ 0 & -e^{i\theta_2} & 0 & -1 & -1 & 3 \end{pmatrix} \quad (6.4.1)$$

with

$$D(\theta_1, \theta_2) = 30[3 - (\cos \theta_1 + \cos \theta_2 + \cos(\theta_1 + \theta_2))]. \quad (6.4.2)$$

Hence,

$$z(\mathcal{L}_{3-12-12}) = \frac{1}{6}[z_{tri} + \ln(15)]. \quad (6.4.3)$$

For a finite 3–12–12 lattice of  $N_1 \times N_2$  cells, the number of spanning trees  $N_{ST}$  is given by (4.12) with  $\Lambda = 2 \times 3^2 \times 5^2 = 450$  and  $\nu = 6$ .

#### 6.5. Union Jack lattice

The Union Jack lattice is the dual of the 4–8–8 lattice. We use (1.17) and (1.23) to obtain

$$N_{ST}(\mathcal{L}_{UJ}) = N_{ST}(\mathcal{L}_{4-8-8}) \quad z(\mathcal{L}_{UJ}) = 2z(\mathcal{L}_{4-8-8}) \quad (6.5.1)$$

where expressions of the right-hand sides in (6.5.1) are given in, respectively, (4.9) and (4.11).

#### 6.6. $N \times \infty$ lattice strips

In this section we give results on the explicit evaluation of  $z$  for  $N \times \infty$  strips of square and triangular lattices for  $2 \leq N \leq 6$  by using (4.18) and (4.19). The results are

$$\begin{aligned} z_{sq(2 \times \infty)} &= \frac{1}{2} \ln(3 + 2\sqrt{2}) = 0.881\,3735\dots \\ z_{sq(3 \times \infty)} &= \frac{2}{3} \ln\left(\frac{1}{2}(5 + \sqrt{21})\right) = 1.044\,5328\dots \\ z_{sq(4 \times \infty)} &= \frac{1}{4} [\ln(3 + 2\sqrt{2}) + 2 \ln(2 + \sqrt{3})] = 1.099\,1657\dots \\ z_{q(5 \times \infty)} &= \frac{2}{5} [\ln(\frac{1}{4}(9 - \sqrt{5} + (70 - 18\sqrt{5})^{1/2})) + \ln(\frac{1}{4}(9 + \sqrt{5} + (70 + 18\sqrt{5})^{1/2}))] \\ &= 1.123\,7289\dots \\ z_{sq(6 \times \infty)} &= \frac{1}{6} [2 \ln(\frac{1}{2}(3 + \sqrt{5})) + 2 \ln(\frac{1}{2}(5 + \sqrt{21})) + \ln(3 + 2\sqrt{2})] \\ &= 1.136\,8654\dots \end{aligned} \quad (6.6.1)$$

and

$$\begin{aligned}
 z_{tri(2 \times \infty)} &= 2 \ln 2 = 1.386\,2943\dots \\
 z_{tri(3 \times \infty)} &= \frac{1}{3}[-\ln 2 + 2 \ln(7 + 3\sqrt{5})] = 1.514\,2805\dots \\
 z_{tri(4 \times \infty)} &= \frac{1}{2}[2 \ln 2 + \ln(3 + \sqrt{7})] = 1.558\,5988\dots \\
 z_{tri(5 \times \infty)} &= \frac{1}{5}[-7 \ln 2 + 2 \ln(13 - \sqrt{5} + (150 - 34\sqrt{5})^{1/2}) \\
 &\quad + 2 \ln(13 + \sqrt{5} + (150 + 34\sqrt{5})^{1/2})] = 1.579\,0412\dots \\
 z_{tri(6 \times \infty)} &= \frac{1}{3}[\ln(5 + \sqrt{13}) + \ln(7 + 3\sqrt{5})] = 1.590\,1339\dots
 \end{aligned} \tag{6.6.2}$$

One observes that the values of  $z$  are monotonically increasing in  $N$ , in accordance with theorem 2.1. One also observes that  $z_{\mathcal{L}(N_2 \times \infty)}$  converge reasonably quickly toward the respective values  $z_{sq}$  in (2.17) and  $z_{tri}$  in (2.19). For example,  $z_{sq(3 \times \infty)}$  and  $z_{sq(6 \times \infty)}$  are, respectively, within 10% and 2.5% of the value (2.17) for the infinite square lattice. Similarly,  $z_{tri(3 \times \infty)}$  and  $z_{tri(6 \times \infty)}$  are, respectively, within 6% and 1.5% of the value (2.19).

### 6.7. Connection with the critical Ising model and dimers

In this section we establish a result relating  $z_{\mathcal{L}}$  for planar lattices  $\mathcal{L}$  to the free energy of the Ising model on  $\mathcal{L}$  at the critical point. We also remark on a connection of spanning trees with dimers for planar lattices. We first state our result as a theorem.

**Theorem 6.1.** *For planar lattices  $\mathcal{L}$  we have the identity*

$$z_{\mathcal{L}} = a_{\mathcal{L}} + 2f_{\mathcal{L}}^c \tag{6.7.1}$$

where  $a_{\mathcal{L}}$  is a lattice-dependent constant, and  $f_{\mathcal{L}}^c$  is the (reduced) free energy of the Ising model on  $\mathcal{L}$  at the critical point.

**Proof.** We use the fact that the Ising model is the infinite bare quartic coupling ( $\lambda \rightarrow \infty$ ) limit of the  $\phi^4$  lattice field theory [25]. The partition function for the  $\phi^4$  quantum field theory is

$$Z = \int_{-\infty}^{\infty} \left[ \prod_i d\phi \right] e^{-S} \tag{6.7.2}$$

where the action  $S$  is an integral of the Lagrangian density with the quadratic part

$$S_{quad} = \frac{1}{2} \int d^2x \left[ \sum_{j=1}^2 \left( \frac{\partial \phi}{\partial x_j} \right)^2 + m^2 \phi^2 \right]. \tag{6.7.3}$$

After integrating by parts, the kinetic terms becomes  $\frac{1}{2} \int d^2x \phi [-\partial^2 + m^2] \phi$ , where  $\partial^2$  is the Laplacian. Further discretizing to a lattice  $\mathcal{L}$ , the integrand in (6.7.3) becomes the summand

$$\frac{1}{2} \sum_{i,j} \phi_i Q_{ij} \phi_j + \frac{1}{2} m^2 \sum_i \phi_i^2 \tag{6.7.4}$$

where  $Q_{ij}$  are elements of the Laplacian matrix  $Q$  given by (1.3).

At the critical point, because the phase transition is of second order with a divergent correlation length, the mass  $m$  in (6.7.4) which is the inverse correlation length vanishes. One is left simply with the term involving  $Q$ . Letting  $\lambda \rightarrow \infty$ , the functional integrals are now reduced to discrete sums over the Ising variables  $\sigma_i = \pm 1$ , and, from the Onsager solution and the correspondence with (4.15), one finds

$$f_{\mathcal{L}}^c = \frac{a_{\mathcal{L}}}{2} + \frac{1}{2\nu} \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_2}{2\pi} \ln D(\theta_1, \theta_2) \tag{6.7.5}$$

where  $a_{\mathcal{L}}$  is a lattice-dependent constant. This establishes (6.7.1) after using (4.16). □

We find the explicit results

$$\begin{aligned}
 z_{sq} &= -\ln 2 + 2f_{sq}^c \\
 z_{tri} &= \ln(\sqrt{3}/2) + 2f_{tri}^c \\
 z_{hc} &= -\ln(2\sqrt{3}) + 2f_{hc}^c \\
 z_{kag} &= \frac{1}{2} \ln 3 - \frac{4}{3} \ln 2 - \frac{1}{3} \ln(2 + \sqrt{3}) + 2f_{kag}^c.
 \end{aligned}
 \tag{6.7.6}$$

The values of  $f_{\mathcal{L}}^c$  in (6.7.6) are well known [26].

Finally, we remark that Temperley [27] has established a bijection between spanning trees on an  $N \times N$  square lattice with free boundaries and dimer configurations on a  $(2N - 1) \times (2N - 1)$  square lattice with one boundary site removed. We shall not repeat the proof here, except to note that this equivalence can be extended more generally to arbitrary planar graphs [28, 29]. Together with theorem 6.1, this bijection implies a connection between dimers and the critical Ising model, a connection which has been observed by Fisher in a weaker form [30].

### 7. Discussion

It is of interest to investigate how close the actual value of  $z_{\mathcal{L}}$  is to the Mckay–Fan–Yau upper bound (2.7). To do this, we define for  $\kappa$ -regular lattices the ratio

$$r_{\mathcal{L}} = \frac{z_{\mathcal{L}}}{\ln C_{\kappa}} \tag{7.1}$$

where  $C_{\kappa}$  is given by (2.6). For lattices which are not  $\kappa$ -regular, we compare  $z_{\mathcal{L}}$  instead with the general upper bound (2.4) and consider the ratio

$$r_{\mathcal{L}} = \frac{z_{\mathcal{L}}}{\ln \kappa_{\text{eff}}}. \tag{7.2}$$

The results are summarized in table 1.

**Table 1.** Values of  $z_{\mathcal{L}}$  and  $r_{\mathcal{L}}$  for different lattices  $\mathcal{L}$ .

$\mathcal{L}$	$d$	$\kappa_{\mathcal{L}}$	$\kappa_{\text{eff}}$	$\nu_{\mathcal{L}}$	$z_{\mathcal{L}}$	$r_{\mathcal{L}}$
3–12–12	2	3		$\frac{1}{2}$	0.720 5633...	0.861
4–8–8 (bathroom-tile)	2	3		$\frac{1}{2}$	0.786 684(1)	0.940
Honeycomb	2	3		$\frac{1}{2}$	0.807 6648...	0.965
Kagomé	2	4		1	1.135 6964...	0.933
Diced	2		4	1	1.135 6964...	0.819
$\mathcal{L}_2$ (square)	2	4		1	1.166 2436...	0.959
Union Jack	2		6	2	1.573 368(2)	0.878
Triangular	2	6		2	1.615 3297...	0.955
$\mathcal{L}_3$ (simple cubic)	3	6		2	1.674 1481(1)	0.990
$\mathcal{L}_{bcc}$ (body-centred cubic)	3	8		3	1.990 2(1)	0.991
$\mathcal{L}_4$	4	8		3	2.000 0(5)	0.996
$\mathcal{L}_5$	5	10		4	2.243	0.998
$\mathcal{L}_6$	6	12		5	2.437	0.999
$\mathcal{L}_{fcc}$ (face-centred cubic)	3	12		5	2.354 4(4)	0.965
$\mathcal{L}_{bcc(4)}$	4	16		7	2.732 3(1)	0.998

As is evident from table 1, for regular lattices that we have studied,  $z_{\mathcal{L}}$  is a monotonically increasing function of the coordination number  $\kappa$ . We also observe that, for a fixed  $\kappa$ , the value of  $z_{\mathcal{L}}$  increases with the spatial dimension of the lattice  $z_{\mathcal{L}}$ . Examples are (a) the triangular and simple cubic lattices (both with  $\kappa = 6$ ), (b) the bcc and  $d = 4$  hypercubic lattices ( $\kappa = 8$ ), although the difference between  $z_{bcc}$  and  $z_{\mathcal{L}_4}$  is very small, and (c) the fcc and  $d = 6$  hypercubic lattices ( $\kappa = 12$ ). Furthermore, our results indicates that in two dimensions the square lattice is a little more densely connected than the kagomé lattice, both of which have  $\kappa = 4$ . The ratio  $r_{\mathcal{L}}$  is observed in several cases to increase with  $\kappa$ , but not in all cases; a counterexample is  $r = 0.959$  for the square lattice ( $\kappa = 4$ ) and  $r = 0.955$  for the triangular lattice ( $\kappa = 6$ ).

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